1. Dissipation in Quantum Mechanics: The Master Equation Approach

1.1 Introduction

This book deals with various quantum-statistical methods and their application to problems in quantum optics. The development of these methods arose out of the need to deal with dissipation in quantum optical systems. Thus, dissipation in quantized systems is a theme unifying the topics covered in the book. Two elementary systems provide the basic building blocks for a number of applications: the damped harmonic oscillator, which describes a single mode of the electromagnetic field in a lossy cavity (a cavity with imperfect mirrors), and the damped two-level atom. The need for a quantized treatment for the damped field mode arose originally in the context of the quantum theory of the maser and the laser. The damped two-level atom is, of course, of very general and fundamental interest, since it is just the problem of spontaneous emission. The book is structured around these two illustrative examples and their use in building quantum-theoretic treatments of resonance fluorescence and the single-mode laser. A second volume will extend the applications to the degenerate parametric oscillator and cavity quantum electrodynamics (cavity QED.). Discussion of the examples will guide the development of fundamental formalism. When we meet such things as master equations, phase-space representations, Fokker-Planck equations and stochastic differential equations, and the related methods of analysis, we will always have a specific application at hand with which to illustrate the formalism. Although formal methods will be introduced essentially from first principles, in places the treatment will necessarily be rather cursory. Ample references to the literature will hopefully offset any deficiencies.

Our objective in this book is to develop the background needed to gain access to issues of current research. The statistical methods we will cover were introduced over approximately two decades beginning in the early 1960's, stimulated by the invention of the laser. They are characterized by an emphasis on the two extremes of statistical physics – the single particle (resonance fluorescence) and very many particles (the single-mode laser). Where possible, they exploit analogies with the methods of classical statistical physics, though the incompatibility of a classical description with quantum mechanics is, in principle, always present. In the second volume we will enter into some

of the modern research topics. The objective there will be to extend the methods discussed in this book, to move away from the one- and many-particle extremes and to face the quantum-classical incompatibility head on.

1.2 Inadequacy of an Ad Hoc Approach

In classical mechanics the essential features of dissipation, namely, the decay of oscillator amplitudes, particle velocities and energies, can be built into the theory by the simple addition of a velocity dependent force. For example, the harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 q^2 \tag{1.1}$$

and equations of motion

$$\dot{q} = p/m, \qquad \dot{p} = -m\omega^2 q, \tag{1.2}$$

becomes a damped harmonic oscillator with the addition of the force $-\gamma p$ to give

$$\dot{q} = p/m, \qquad \dot{p} = -\gamma p - m\omega^2 q, \tag{1.3}$$

or the familiar equation

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0. \tag{1.4}$$

Can we simply transfer this approach to the quantized harmonic oscillator? For the quantized oscillator q and p become operators, $\hat{q}$ and $\hat{p}$, and (1.2) gives the Heisenberg equations of motion obtained from Hamiltonian (1.1) via the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar. \tag{1.5}$$

After adding $-\gamma p$ to (1.3), the equations of motion remain linear; thus, the classical solution still holds when q and p become operators, and the expectation values of $\hat{q}$ and $\hat{p}$ will be damped in the same way as the classical variables. We seem to be in good shape. Consider, however, the evolution of the commutator $[\hat{q}, \hat{p}]$. From (1.3)

$$\begin{aligned}\frac{d}{dt}[\hat{q}, \hat{p}] &= \dot{\hat{q}}\hat{p} + \hat{q}\dot{\hat{p}} - \dot{\hat{p}}\hat{q} - \hat{p}\dot{\hat{q}} \\ &= -\gamma[\hat{q}, \hat{p}],\end{aligned}$$

and

$$[\hat{q}(t), \hat{p}(t)] = e^{-\gamma t}[\hat{q}(0), \hat{p}(0)] = e^{-\gamma t} i\hbar. \tag{1.6}$$

As a consequence of this decay of the commutator the Heisenberg uncertainty also decays; the Heisenberg uncertainty relation becomes

$$\Delta q \Delta p \geq \tfrac{1}{2}\hbar e^{-\gamma t}. \tag{1.7}$$

In the face of this difficulty there have been various attempts to consistently incorporate dissipation into quantum mechanics. Some approaches based on novel quantization procedures remain controversial. We will not review these issues here. Of course, in many of the traditional domains of quantum mechanics dissipation plays no role: in the analysis of atomic structure, or the calculation of harmonic oscillator eigenstates and the like. The situation is quite different, though, in quantum optics. For example, the phenomenon of laser action, which gave birth to this field, takes place in a *lossy* cavity. In fact, applications in quantum optics have played a central role in developing methods to treat quantum-mechanical dissipation. We follow the widely accepted approach pioneered by Senitzky [1.1] for describing lossy maser cavities. Some discussion of alternative points of view can be found in papers by Ray [1.2] and Caldeira and Leggett [1.3], and references therein.

1.3 System Plus Reservoir Approach

The system plus reservoir approach begins from a microscopic view of the mechanism underlying dissipation. Although the procedure leading to (1.3) and (1.4) is often adequate in classical mechanics, even there it provides an incomplete description. In particular, equations (1.2) are time-reversal invariant, while in (1.3) this symmetry has been broken. If we want to understand the origin of this irreversibility we must begin by recognizing that the oscillator is damped through interactions with a large and complex system – its environment. This recognition also leads us to the fundamental relationship between dissipation and fluctuations. If the environment is some large system in thermal equilibrium, it will exert a fluctuating force $F(t)$ on an oscillator coupled to it, in addition to soaking up the oscillator's energy. Equation (1.4) must generally be replaced by a stochastic equation

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = F(t)/m. \tag{1.8}$$

In many situations the added noise source cannot be overlooked – in electrical circuits, for example.

We observe that damping takes place through the coupling of the damped system to its environment. Is there anything in this observation to suggest a resolution of our problem with commutators? Well, the interaction between systems mixes their operators in a way which certainly does play a role in preserving commutators in time. Consider resonant harmonic oscillators coupled in the rotating-wave approximation. The Hamiltonian is

$$H = \hbar\omega a^\dagger a + \hbar\omega b^\dagger b + \hbar\kappa(a^\dagger b + a b^\dagger), \tag{1.9}$$

where ω is the frequency of the oscillators, κ is a coupling constant, $a^\dagger$ and $b^\dagger$ are creation operators, and a and b are the corresponding annihilation operators, satisfying commutation relations

$$[a, a^\dagger] = 1, \qquad [b, b^\dagger] = 1. \tag{1.10}$$

Note 1.1 To understand the origin of the Hamiltonian (1.9) first note that the free oscillator Hamiltonian (1.1) becomes

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \tag{1.11}$$

where $\frac{1}{2}\hbar\omega$ is the zero-point energy, under the transformation

$$a \equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} + i\hat{p}), \tag{1.12a}$$

$$a^\dagger \equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} - i\hat{p}). \tag{1.12b}$$

Then (1.6) becomes

$$[a, a^\dagger] = e^{-\gamma t}. \tag{1.13}$$

In the *rotating-wave approximation* an interaction energy proportional to $\hat{q}_a\hat{q}_b$ gives the interaction Hamiltonian $\hbar\kappa(a^\dagger b + a b^\dagger)$ after the highly oscillatory terms (energy nonconserving terms) ab and $a^\dagger b^\dagger$ are neglected.

The solutions to the Heisenberg equations of motion following from (1.9) are

$$a(t) = e^{-i\omega t}[a(0)\cos\kappa t - ib(0)\sin\kappa t], \tag{1.14a}$$

$$b(t) = e^{-i\omega t}[b(0)\cos\kappa t - ia(0)\sin\kappa t]. \tag{1.14b}$$

Then

$$[a(t), a^\dagger(t)] = [a(0), a^\dagger(0)]\cos^2\kappa t + [b(0), b^\dagger(0)]\sin^2\kappa t = 1. \tag{1.15}$$

We see that the commutator for $a(t)$ and $a^\dagger(t)$ is preserved in time only by the presence of the operator $b(0)$ mixed into the solution for $a(t)$. Taking the environmental interaction into account in the treatment of dissipation, we might anticipate a similar mixing of environmental operators into the operators of the damped system in such a way as to preserve commutation relations. This is precisely what Senitzky found [1.1]. The fluctuating force in (1.8) becomes an operator in Senitzky's theory. Contributions from this environmental operator in the solutions for $\hat{q}(t)$ and $\hat{p}(t)$ introduce thermal fluctuations, and also preserve the commutation relations.

The master equation method we now discuss is essentially a Schrödinger picture version of Senitzky's theory. It is somewhat less transparent on this point about preserving commutation relations, so it is valuable to study Senitzky's calculation in the Heisenberg picture as well as the following. In both

the philosophy is to model environmental interactions by coupling the undamped system S to a reservoir R, beginning with a Hamiltonian in the general form

$$H = H_S + H_R + H_{SR}, \tag{1.16}$$

where H_S and H_R are Hamiltonians for S and R, respectively, and H_{SR} is an interaction Hamiltonian. The reservoir is only of indirect interest, and its properties need only be specified in very general terms; for example, by a temperature and an energy density of states. For illustrative purposes we will give H_R and H_{SR} an explicit form once we get a little further into the calculation.

The derivation given here follows the treatments by Louisell [1.4] and Haken [1.5] fairly closely. There are some minor differences in the way approximations are introduced, and no attempt is made to follow either author's notation. A rather different and more specialized approach is taken by Sargent, Scully and Lamb [1.6]. These authors get away without having to deal with the complicated frequency and time integrals we will meet in our calculation. It is a useful exercise to study their calculation and try to find where they introduce the physical assumptions we will use to deal with these integrals. The physics must, of course, be the same.

We are seeking information about the system S without requiring detailed information about the composite system $S \oplus R$. We will let $\chi(t)$ be the density operator for $S \oplus R$ and define the reduced density operator $\rho(t)$ by

$$\rho(t) \equiv \mathrm{tr}_R[\chi(t)], \tag{1.17}$$

where the trace is taken over the reservoir states. Clearly, if $\hat{O}$ is an operator in the Hilbert space of S we can calculate its average in the Schrödinger picture if we have knowledge of $\rho(t)$ alone, and not of the full $\chi(t)$:

$$\langle \hat{O} \rangle = \mathrm{tr}_{S\oplus R}[\hat{O}\chi(t)] = \mathrm{tr}_S\{\hat{O}\mathrm{tr}_R[\chi(t)]\} = \mathrm{tr}_S[\hat{O}\rho(t)]. \tag{1.18}$$

Our objective is to obtain an equation for $\rho(t)$ with the properties of R entering only as parameters.

1.3.1 The Schrödinger Equation in Integro-Differential Form

The Schrödinger equation for χ reads

$$\dot{\chi} = \frac{1}{i\hbar}[H, \chi], \tag{1.19}$$

where H is given by (1.16). We transform (1.19) into the interaction picture, separating the rapid motion generated by $H_S + H_R$ from the slow motion generated by the interaction H_{SR}. Defining

$$\tilde{\chi}(t) \equiv e^{(i/\hbar)(H_S+H_R)t}\chi(t)e^{-(i/\hbar)(H_S+H_R)t}, \tag{1.20}$$

from (1.16) and (1.19), we obtain

$$\begin{aligned}\dot{\tilde{\chi}} &= \frac{i}{\hbar}(H_S + H_R)\tilde{\chi} - \frac{i}{\hbar}\tilde{\chi}(H_S + H_R) + e^{(i/\hbar)(H_S+H_R)t}\dot{\chi}e^{-(i/\hbar)(H_S+H_R)t} \\ &= \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \tilde{\chi}],\end{aligned} \tag{1.21}$$

where $\tilde{H}_{SR}(t)$ is explicitly time-dependent:

$$\tilde{H}_{SR}(t) \equiv e^{(i/\hbar)(H_S+H_R)t}H_{SR}e^{-(i/\hbar)(H_S+H_R)t}. \tag{1.22}$$

We now integrate (1.21) formally to give

$$\tilde{\chi}(t) = \chi(0) + \frac{1}{i\hbar}\int_0^t dt'\,[\tilde{H}_{SR}(t'), \tilde{\chi}(t')], \tag{1.23}$$

and substitute for $\tilde{\chi}(t)$ inside the commutator in (1.21):

$$\dot{\tilde{\chi}} = \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \chi(0)] - \frac{1}{\hbar^2}\int_0^t dt'\,[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]]. \tag{1.24}$$

This equation is exact. Equation (1.19) has simply been cast into a convenient form which helps us identify reasonable approximations.

1.3.2 Born and Markov Approximations

We will assume that the interaction is turned on at $t = 0$ and that no correlations exist between S and R at this initial time. Then $\chi(0) = \tilde{\chi}(0)$ factorizes as

$$\chi(0) = \rho(0)R_0, \tag{1.25}$$

where R_0 is an initial reservoir density operator. Then, noting that

$$\mathrm{tr}_R[\tilde{\chi}(t)] = e^{(i/\hbar)H_S t}\rho(t)e^{-(i/\hbar)H_S t} \equiv \tilde{\rho}(t), \tag{1.26}$$

after tracing over the reservoir, (1.24) gives the *master equation*

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{tr}_R\{[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]]\}, \tag{1.27}$$

where, for simplicity, we have eliminated the term $(1/i\hbar)\mathrm{tr}_R\{[\tilde{H}_{SR}(t), \chi(0)]\}$ with the assumption

$$\mathrm{tr}_R[\tilde{H}_{SR}(t)R_0] = 0. \tag{1.28}$$

This is guaranteed if the reservoir operators coupling to S have zero mean in the state R_0, a condition which can always be arranged by simply including $\mathrm{tr}_R(H_{SR}R_0)$ in the system Hamiltonian (see Sect. 2.2.4 and Note 8.8).

We have stated that $\tilde{\chi}$ factorizes at $t = 0$. At later times correlations between S and R will arise due to the coupling between the system and

the reservoir. We have assumed, however, that this coupling is very weak, and at all times $\chi(t)$ should only show deviations of order H_{SR} from an uncorrelated state. Furthermore, R is a large system whose state should be virtually unaffected by its coupling to S (of course, we expect the state of S to be significantly affected by R – we want it to be damped). We therefore write

$$\tilde{\chi}(t) = \tilde{\rho}(t)R_0 + O(H_{SR}). \tag{1.29}$$

Now we can make our first major approximation, a *Born approximation*. Neglecting terms higher than second order in H_{SR}, we write (1.27) as

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t dt' \,\mathrm{tr}_R\{[\tilde{H}_{SR}(t),[\tilde{H}_{SR}(t'),\tilde{\rho}(t')R_0]]\}. \tag{1.30}$$

A detailed discussion of this approximation can be found in the work of Haake [1.7, 1.8].

Equation (1.30) is still a complicated equation. In particular, it is not Markovian since the future evolution of $\tilde{\rho}(t)$ depends on its past history through the integration over $\tilde{\rho}(t')$ (the future behavior of a Markovian system depends only on its present state). Our second major approximation, the *Markov approximation*, replaces $\tilde{\rho}(t')$ by $\tilde{\rho}(t)$ to obtain a *master equation in the Born-Markov approximation*:

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t dt' \,\mathrm{tr}_R\{[\tilde{H}_{SR}(t),[\tilde{H}_{SR}(t'),\tilde{\rho}(t)R_0]]\}. \tag{1.31}$$

1.3.3 The Markov Approximation and Reservoir Correlations

Markovian behavior seems reasonable on physical grounds. Potentially, S can depend on its past history because its earlier states become imprinted as changes in the reservoir state through the interaction H_{SR}; earlier states are then reflected back on the future evolution of S as it interacts with the changed reservoir. If, however, the reservoir is a large system maintained in thermal equilibrium, we do not expect it to preserve the minor changes brought by its interaction with S for very long; not for long enough to significantly affect the future evolution of S. It becomes a question of reservoir correlation time versus the time scale for significant change in S. By studying the integrand of (1.30) with this view in mind we can make the underlying assumption of the Markov approximation more explicit.

Let us make our model a little more specific by writing

$$H_{SR} = \hbar\sum_i s_i\Gamma_i, \tag{1.32}$$

where the s_i are operators in the Hilbert space of S and the Γ_i are reservoir operators, operators in the Hilbert space of R. Then

$$\begin{aligned}\tilde{H}_{SR}(t) &= \hbar\sum_i e^{(i/\hbar)(H_S+H_R)t}s_i\Gamma_i e^{-(i/\hbar)(H_S+H_R)t}\\ &= \hbar\sum_i\left(e^{(i/\hbar)H_St}s_ie^{-(i/\hbar)H_St}\right)\left(e^{(i/\hbar)H_Rt}\Gamma_ie^{-(i/\hbar)H_Rt}\right)\\ &= \hbar\sum_i\tilde{s}_i(t)\tilde{\Gamma}_i(t).\end{aligned} \tag{1.33}$$

The master equation in the Born approximation [Eq. (1.30)] is now

$$\begin{aligned}\dot{\tilde{\rho}} &= -\sum_{i,j}\int_0^t dt'\,\mathrm{tr}_R\{[\tilde{s}_i(t)\tilde{\Gamma}_i(t),[\tilde{s}_j(t')\tilde{\Gamma}_j(t'),\tilde{\rho}(t')R_0]]\}\\ &= -\sum_{i,j}\int_0^t dt'\Big\{\tilde{s}_i(t)\tilde{s}_j(t')\tilde{\rho}(t')\,\mathrm{tr}_R[\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')R_0]\\ &\quad -\tilde{s}_i(t)\tilde{\rho}(t')\tilde{s}_j(t')\mathrm{tr}_R[\tilde{\Gamma}_i(t)R_0\tilde{\Gamma}_j(t')] - \tilde{s}_j(t')\tilde{\rho}(t')\tilde{s}_i(t)\\ &\quad\times\mathrm{tr}_R[\tilde{\Gamma}_j(t')R_0\tilde{\Gamma}_i(t)] + \tilde{\rho}(t')\tilde{s}_j(t')\tilde{s}_i(t)\,\mathrm{tr}_R[R_0\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)]\Big\}\\ &= -\sum_{i,j}\int_0^t dt'\Big\{[\tilde{s}_i(t)\tilde{s}_j(t')\tilde{\rho}(t') - \tilde{s}_j(t')\tilde{\rho}(t')\tilde{s}_i(t)]\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R\\ &\quad + [\tilde{\rho}(t')\tilde{s}_j(t')\tilde{s}_i(t) - \tilde{s}_i(t)\tilde{\rho}(t')\tilde{s}_j(t')]\langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_R\Big\},\end{aligned} \tag{1.34}$$

where we have used the cyclic property of the trace – $\mathrm{tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{tr}(\hat{C}\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{C}\hat{A})$ – and write

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R = \mathrm{tr}_R[R_0\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')], \tag{1.35a}$$

$$\langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_R = \mathrm{tr}_R[R_0\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)]. \tag{1.35b}$$

The properties of the reservoir enter (1.34) through the two correlation functions (1.35a) and (1.35b). We can justify the replacement of $\tilde{\rho}(t')$ by $\tilde{\rho}(t)$ if these correlation functions decay very rapidly on the timescale on which $\tilde{\rho}(t)$ varies. Ideally, we might take

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R \propto \delta(t-t'). \tag{1.36}$$

The Markov approximation then relies, as suggested, on the existence of two widely separated time scales: a slow time scale for the dynamics of the system S, and a fast time scale characterizing the decay of reservoir correlation functions. Further discussion of this point is given by Schieve and Middleton [1.9]. We will look explicitly at reservoir correlation functions and the separation of time scales in our first example.

1.4 The Damped Harmonic Oscillator

1.4.1 Master Equation for the Damped Harmonic Oscillator

We now adopt an explicit model. For the Hamiltonian of the composite system $S \oplus R$ we write

$$H_S \equiv \hbar\omega_0 a^\dagger a, \tag{1.37a}$$

$$H_R \equiv \sum_j \hbar\omega_j {r_j}^\dagger r_j, \tag{1.37b}$$

$$H_{SR} \equiv \sum_j \hbar(\kappa_j^* a {r_j}^\dagger + \kappa_j a^\dagger r_j) = \hbar(a\Gamma^\dagger + a^\dagger \Gamma). \tag{1.37c}$$

The system S is an harmonic oscillator with frequency ω_0 and creation and annihilation operators $a^\dagger$ and a, respectively; the reservoir R is modeled as a collection of harmonic oscillators with frequencies ω_j, and corresponding creation and annihilation operators ${r_j}^\dagger$ and r_j, respectively; the oscillator a couples to the jth reservoir oscillator via a coupling constant κ_j in the rotating-wave approximation. We take the reservoir to be in thermal equilibrium at temperature T, with density operator

$$R_0 = \prod_j e^{-\hbar\omega_j {r_j}^\dagger r_j/k_B T}\left(1 - e^{-\hbar\omega_j/k_B T}\right), \tag{1.38}$$

where k_B is Boltzmann's constant. It is not necessary to be so specific about the reservoir model. Haken [1.5], for example, keeps his discussion quite general. Aside, however, from its pedagogical clarity, the oscillator model is physically reasonable in many circumstances. The reservoir oscillators might be the many modes of the vacuum radiation field into which an optical cavity mode decays through partially transmitting mirrors, or into which an excited atom decays via spontaneous emission; alternatively, they might represent phonon modes in a solid.

The identification with (1.34) is made by setting

$$s_1 = a, \qquad s_2 = a^\dagger, \tag{1.39a}$$

$$\Gamma_1 = \Gamma^\dagger \equiv \sum_j \kappa_j^* {r_j}^\dagger, \qquad \Gamma_2 = \Gamma \equiv \sum_j \kappa_j r_j, \tag{1.39b}$$

and then from (1.33) and (1.37), the operators in the interaction picture are

$$\tilde{s}_1(t) = e^{i\omega_0 a^\dagger a t} a e^{-i\omega_0 a^\dagger a t} = a e^{-i\omega_0 t}, \tag{1.40a}$$

$$\tilde{s}_2(t) = e^{i\omega_0 a^\dagger a t} a^\dagger e^{-i\omega_0 a^\dagger a t} = a^\dagger e^{i\omega_0 t}, \tag{1.40b}$$

and

$$\tilde{\Gamma}_1(t) = \tilde{\Gamma}^\dagger(t) = \exp\left(i\sum_n \omega_n {r_n}^\dagger r_n t\right)\sum_j \kappa_j^* {r_j}^\dagger \exp\left(-i\sum_m \omega_m {r_m}^\dagger r_m t\right)$$
$$= \sum_j \kappa_j^* {r_j}^\dagger e^{i\omega_j t}, \tag{1.41a}$$

$$\tilde{\Gamma}_2(t) = \tilde{\Gamma}(t) = \exp\left(i\sum_n \omega_n {r_n}^\dagger r_n t\right)\sum_j \kappa_j r_j \exp\left(-i\sum_m \omega_m {r_m}^\dagger r_m t\right)$$
$$= \sum_j \kappa_j r_j e^{-i\omega_j t}, \tag{1.41b}$$

where in (1.41) we use the fact that operators for different reservoir oscillators commute. To show, for example, that $e^{i\omega_0 a^\dagger a t} a e^{-i\omega_0 a^\dagger a t} = a e^{-i\omega_0 t}$, observe that the left hand side is just the formal solution to the Heisenberg equation of motion $\dot{a} = -i\omega_0[a, a^\dagger a] = -i\omega_0 a$. Note that, from (1.38) and (1.41), $\langle\tilde{\Gamma}_1(t)\rangle_R = \langle\tilde{\Gamma}_2(t)\rangle_R = 0$, as required by the assumption (1.28).

Now, since the summation in (1.34) runs over $i = 1, 2$ and $j = 1, 2$, the integrand involves sixteen terms. We write

$$\dot{\tilde{\rho}} = -\int_0^t dt' \Big\{\left[aa\tilde{\rho}(t') - a\tilde{\rho}(t')a\right]e^{-i\omega_0(t+t')}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}^\dagger(t')\rangle_R + \text{h.c.}$$
$$+\left[a^\dagger a^\dagger\tilde{\rho}(t') - a^\dagger\tilde{\rho}(t')a^\dagger\right]e^{i\omega_0(t+t')}\langle\tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle_R + \text{h.c.}$$
$$+\left[aa^\dagger\tilde{\rho}(t') - a^\dagger\tilde{\rho}(t')a\right]e^{-i\omega_0(t-t')}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t')\rangle_R + \text{h.c.}$$
$$+\left[a^\dagger a\tilde{\rho}(t') - a\tilde{\rho}(t')a^\dagger\right]e^{i\omega_0(t-t')}\langle\tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\rangle_R + \text{h.c.}\Big\}, \tag{1.42}$$

where the reservoir correlation functions are explicitly:

$$\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}^\dagger(t')\rangle_R = \sum_{j,k}\kappa_j^*\kappa_k^* e^{i\omega_j t}e^{i\omega_k t'}\,\text{tr}_R(R_0 {r_j}^\dagger {r_k}^\dagger) = 0, \tag{1.43}$$

$$\langle\tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle_R = \sum_{j,k}\kappa_j\kappa_k e^{-i\omega_j t}e^{-i\omega_k t'}\,\text{tr}_R(R_0 r_j r_k) = 0, \tag{1.44}$$

$$\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t')\rangle_R = \sum_{j,k}\kappa_j^*\kappa_k e^{i\omega_j t}e^{-i\omega_k t'}\,\text{tr}_R(R_0 {r_j}^\dagger r_k)$$
$$= \sum_j |\kappa_j|^2 e^{i\omega_j(t-t')}\bar{n}(\omega_j, T), \tag{1.45}$$

$$\langle\tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\rangle_R = \sum_{j,k}\kappa_j\kappa_k^* e^{-i\omega_j t}e^{i\omega_k t'}\,\text{tr}_R(R_0 r_j {r_k}^\dagger)$$
$$= \sum_j |\kappa_j|^2 e^{-i\omega_j(t-t')}[\bar{n}(\omega_j, T) + 1], \tag{1.46}$$

with

$$\bar{n}(\omega_j, T) = \mathrm{tr}_R(R_0 r_j{}^\dagger r_j) = \frac{e^{-\hbar\omega_j/k_BT}}{1 - e^{-\hbar\omega_j/k_BT}}. \tag{1.47}$$

The correlation functions (1.43)–(1.46) follow quite readily by evaluating the trace using the multimode Fock states as a basis. $\bar{n}(\omega_j, T)$ is the mean photon number for an oscillator with frequency ω_j in thermal equilibrium at temperature T.

The nonvanishing reservoir correlation functions (1.45) and (1.46) involve a summation over the reservoir oscillators. We change this summation to an integration by introducing a density of states $g(\omega)$ such that $g(\omega)d\omega$ gives the number of oscillators with frequencies in the interval ω to $\omega + d\omega$. Making the change of variable

$$\tau = t - t', \tag{1.48}$$

(1.42) can then be restated as

$$\dot{\tilde{\rho}} = -\int_0^t d\tau \Big\{ \Big[a a^\dagger \tilde{\rho}(t-\tau) - a^\dagger \tilde{\rho}(t-\tau) a \Big] e^{-i\omega_0\tau} \langle \tilde{\Gamma}^\dagger(t) \tilde{\Gamma}(t-\tau) \rangle_R + \text{h.c.} + \Big[a^\dagger a \tilde{\rho}(t-\tau) - a \tilde{\rho}(t-\tau) a^\dagger \Big] e^{i\omega_0\tau} \langle \tilde{\Gamma}(t) \tilde{\Gamma}^\dagger(t-\tau) \rangle_R + \text{h.c.} \Big\}, \tag{1.49}$$

where the nonzero reservoir correlation functions are

$$\langle \tilde{\Gamma}^\dagger(t) \tilde{\Gamma}(t-\tau) \rangle_R = \int_0^\infty d\omega\, e^{i\omega\tau} g(\omega) |\kappa(\omega)|^2 \bar{n}(\omega, T), \tag{1.50}$$

$$\langle \tilde{\Gamma}(t) \tilde{\Gamma}^\dagger(t-\tau) \rangle_R = \int_0^\infty d\omega\, e^{-i\omega\tau} g(\omega) |\kappa(\omega)|^2 [\bar{n}(\omega, T) + 1], \tag{1.51}$$

with

$$\bar{n}(\omega, T) = \frac{e^{-\hbar\omega/k_BT}}{1 - e^{-\hbar\omega/k_BT}}. \tag{1.52}$$

We can now argue more specifically about the Markov approximation. Are (1.50) and (1.51) approximately proportional to $\delta(\tau)$? We can certainly see that for τ "large enough" the oscillating exponential will average the "slowly varying" functions $g(\omega)$, $|\kappa(\omega)|^2$, and $\bar{n}(\omega, T)$ essentially to zero. However, how large is large enough? Can we get some idea of the width of these correlation functions? Let us look at (1.50), taking $g(\omega)|\kappa(\omega)|^2 = C\omega$, with C a constant. This correlation function may be evaluated in terms of the trigamma function [1.10]:

$$\begin{aligned} \langle \tilde{\Gamma}^\dagger(t) \tilde{\Gamma}(t-\tau) \rangle_R &= C \int_0^\infty d\omega\, e^{i\omega\tau} \frac{\omega e^{-\hbar\omega/k_BT}}{1 - e^{-\hbar\omega/k_BT}} \\ &= C t_R^{-2} \int_0^\infty dx\, \frac{x e^{-(1 - i\tau/t_R)x}}{1 - e^{-x}} \\ &= C t_R^{-2} \psi'(1 - i\tau/t_R), \end{aligned} \tag{1.53}$$

where we have defined the reservoir correlation time $t_R = \hbar/k_BT$. A simple approximation gives some insight into the behavior of the trigamma function. Set

$$\frac{\omega e^{-\hbar\omega/k_BT}}{1 - e^{-\hbar\omega/k_BT}} \approx \frac{k_BT}{\hbar} e^{-\hbar\omega/k_BT}; \tag{1.54}$$

then

$$\begin{aligned} \langle \tilde{\Gamma}^\dagger(t) \tilde{\Gamma}(t-\tau) \rangle_R &\approx C \frac{k_BT}{\hbar} \int_0^\infty d\omega\, e^{i\omega\tau} e^{-\hbar\omega/k_BT} \\ &\approx C t_R^{-2} \frac{1 + i\tau/t_R}{1 + (\tau/t_R)^2}. \end{aligned} \tag{1.55}$$

The approximation is accurate for low frequencies, but is not so good for $\omega \sim k_BT/\hbar = t_R^{-1}$; here the error is $\sim 40\%$. It is adequate, nevertheless, to give us a feel for the qualitative behavior of the reservoir correlation function. Actually, the exact result for the *real* part of the correlation function can be computed with little effort using the formula [1.10]

$$\mathrm{Re}[\psi'(1 - i\tau/t_R)] = \tfrac{1}{2}\pi^2 [1 - \coth^2(\pi\tau/t_R)] + \tfrac{1}{2}(\tau/t_R)^{-2}. \tag{1.56}$$

The exact result is plotted together with the real part of (1.55) for comparison in Fig. 1.1(a).

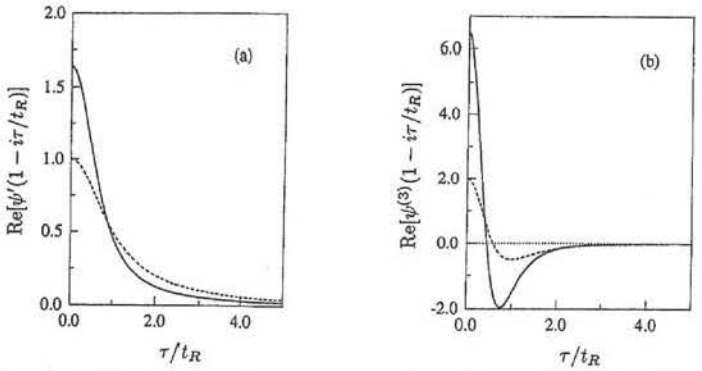

Fig. 1.1 (a) Real part of the reservoir correlation function for $g(\omega)|\kappa(\omega)|^2 = C\omega$ plotted from (1.56) (solid line) and (1.55) (dashed line). (b) Real part of the reservoir correlation function for $g(\omega)|\kappa(\omega)|^2 = C\omega^3$ plotted from (1.61) (solid line) and (1.60) (dashed line).

Equation (1.55) indicates a correlation function peaked about $\tau = 0$ with a width $t_R = \hbar/k_BT$. In (1.49) the reservoir correlation functions are integrated against two time-dependent terms: $\tilde{\rho}(t-\tau)$ and $e^{\pm i\omega_0\tau}$. Now, at room

temperature $\hbar/k_BT \approx 0.25 \times 10^{-13}$s. If the oscillator a represents an optical cavity mode, we expect $\tilde{\rho}(t-\tau)$ to vary on the time scale of a typical cavity decay time, $t_S \sim 10^{-8}$s; and if ω_0 is an optical frequency, $e^{\pm i\omega_0\tau}$ oscillates on a time scale $t_0 \sim 10^{-15}$s. Then, since $t_S/t_R \sim 10^5$ it seems we can justify the Markov approximation and replace $\tilde{\rho}(t-\tau)$ by $\tilde{\rho}(t)$. But, with $t_0/t_R \sim 10^{-2}$, we cannot set $\tau = 0$ in the terms $e^{\pm i\omega_0\tau}$. Rather, integrating the reservoir correlation functions against these oscillating terms will extract the ω_0 frequency component of the correlation functions, as in a Fourier transform.

After taking a closer look we might be a little worried about the imaginary part of (1.55). This has a long tail which decays as $(\tau/t_R)^{-1}$; the integral of this tail is logarithmically divergent; far out in the tail the replacement of $\tilde{\rho}(t-\tau)$ by $\tilde{\rho}(t)$ will not be justified. It is, however, the ω_0 frequency component of the product $\tilde{\rho}(t-\tau)\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R$ that survives the integral in (1.49), and with $t_0 < t_R << t_S$ this frequency component is contributed by the short-time behavior of (1.55) where the replacement of $\tilde{\rho}(t-\tau)$ by $\tilde{\rho}(t)$ is justified.

In fact, the divergent tail is a consequence of the form we have chosen for $g(\omega)\kappa^2(\omega)$. More generally, if we take $g(\omega)|\kappa(\omega)|^2 = C\omega^n$, with n a positive integer,

$$\begin{aligned}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R &= (-i)^{n-1}\frac{d^{n-1}}{d\tau^{n-1}}\left[Ct_R^{-2}\psi'(1-i\tau/t_R)\right] \\ &= Ct_R^{-(n+1)}(-1)^{n-1}\psi^{(n)}(1-i\tau/t_R), \end{aligned} \tag{1.57}$$

where the $\psi^{(n)}$ are the polygamma functions [2.10]. In the approximation (1.54)

$$\begin{aligned}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R &= (-i)^{n-1}\frac{d^{n-1}}{d\tau^{n-1}}\left[Ct_R^{-2}\frac{1+i\tau/t_R}{1+(\tau/t_R)^2}\right] \\ &= Ct_R^{-(n+1)}(-i)^{n-1}\frac{d^{n-1}}{d(\tau/t_R)^{n-1}}\frac{1+i\tau/t_R}{1+(\tau/t_R)^2}. \end{aligned} \tag{1.58}$$

For $\tau/t_R >> 1$ the asymptotic form of the polygamma function gives

$$\begin{aligned}\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R \sim -Ct_R^{-(n+1)}&\left[i^{n+1}(n-1)!\right]\Big[\tfrac{1}{2}n(\tau/t_R)^{-(n+1)} \\ &- i(\tau/t_R)^{-n}\Big], \end{aligned} \tag{1.59}$$

which has no $(\tau/t_R)^{-1}$ tail for $n > 1$.

The case $n = 3$ is of special interest since this corresponds to the form of $g(\omega)|\kappa(\omega)|^2$ that we will meet when we apply our theory to the damped two-level atom (Sect. 2.2). The approximate result (1.58) gives

$$\langle\tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t-\tau)\rangle_R \approx Ct_R^{-4}\frac{2\left[1-3(\tau/t_R)^2\right]+i2(\tau/t_R)\left[3-(\tau/t_R)^2\right]}{\left[1+(\tau/t_R)^2\right]^3}. \tag{1.60}$$

For comparison with the real part of this result, the real part of the exact correlation function can be computed from (1.57) using the formula

$$\begin{aligned}\mathrm{Re}\left[\psi^{(3)}(1-i\tau/t_R)\right] &= \pi^4\left[1-\coth^2(\pi\tau/t_R)\right]\left[1-3\coth^2(\pi\tau/t_R)\right] \\ &\quad - 3(\tau/t_R)^{-4}. \end{aligned} \tag{1.61}$$

This formula is obtained by taking two derivatives of (1.56). The exact and approximate results for the real part of the correlation function are plotted in Fig. 1.1(b). Again the correlation function is peaked around $\tau = 0$ with a width $\sim t_R$. The approximate correlation function (1.60) explicitly shows the $(\tau/t_R)^{-4}$ and $(\tau/t_R)^{-3}$ dependence for the real and imaginary parts, respectively, in the large τ limit, as given by (1.59).

Exercise 1.1 Consider the correlation function (1.51). The second term inside the square bracket comes from quantum (vacuum) fluctuations. It arose from our use of the boson commutation relation in the derivation of (1.46). What contribution does this term make to the correlation function?

Continuing our derivation now from (1.49), it is actually more straightforward to evaluate the time integral first, without performing the frequency integrals to obtain an explicit form for the reservoir correlation functions. This is possible now we are satisfied that the τ integration is dominated by times that are much shorter than the time scale for the evolution of $\tilde{\rho}$. With $\tilde{\rho}(t-\tau)$ replaced by $\tilde{\rho}(t)$ (Markov approximation), (1.49) becomes

$$\dot{\tilde{\rho}} = \alpha(a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho}) + \beta(a\tilde{\rho}a^\dagger + a^\dagger\tilde{\rho}a - a^\dagger a\tilde{\rho} - \tilde{\rho}aa^\dagger) + \text{h.c.}, \tag{1.62}$$

with

$$\alpha \equiv \int_0^t d\tau\int_0^\infty d\omega\, e^{-i(\omega-\omega_0)\tau}g(\omega)|\kappa(\omega)|^2, \tag{1.63}$$

$$\beta \equiv \int_0^t d\tau\int_0^\infty d\omega\, e^{-i(\omega-\omega_0)\tau}g(\omega)|\kappa(\omega)|^2\bar{n}(\omega,T). \tag{1.64}$$

Then, since t is of the order of t_S and the τ integration is dominated by much shorter times $\sim t_R$, we can extend the τ integration to infinity and evaluate α and β using

$$\lim_{t\to\infty}\int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau} = \pi\delta(\omega-\omega_0) + i\frac{P}{\omega_0-\omega}, \tag{1.65}$$

where P indicates the Cauchy principal value. We find

$$\alpha = \pi g(\omega_0)|\kappa(\omega_0)|^2 + i\Delta, \tag{1.66}$$

$$\beta = \pi g(\omega_0)|\kappa(\omega_0)|^2\bar{n}(\omega_0) + i\Delta', \tag{1.67}$$

with

$$\Delta \equiv P\int_0^\infty d\omega\,\frac{g(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}, \tag{1.68}$$

$$\Delta' \equiv P\int_0^\infty d\omega\,\frac{g(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}\bar{n}(\omega,T). \tag{1.69}$$

Note 1.2 To obtain (1.65), we have

$$\int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau} = \frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} - i\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0}.$$

The limit as t tends to infinity is defined anticipating the role of the right-hand side inside an integration over ω, thus:

$$\begin{aligned}\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} &= f(\omega_0)\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\,\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0}\\ &= \pi f(\omega_0)\\ &= \int_{-\infty}^{\infty} d\omega\,\pi\delta(\omega-\omega_0)f(\omega);\end{aligned}$$

also

$$\begin{aligned}\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega f(\omega)&\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0}\\ &= \int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega-\omega_0} - \lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)\cos(\omega-\omega_0)t}{\omega-\omega_0}\\ &= P\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega-\omega_0},\end{aligned}$$

where the term

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)\cos(\omega-\omega_0)t}{\omega-\omega_0}$$

subtracts the singularity at $\omega=\omega_0$ to give the principal value integral [1.11].

We finally have our master equation for the damped harmonic oscillator. After defining

$$\gamma \equiv 2\pi g(\omega_0)|\kappa(\omega_0)|^2, \tag{1.70a}$$

$$\bar{n} \equiv \bar{n}(\omega_0,T), \tag{1.70b}$$

from (1.62), (1.66), and (1.67), we obtain

$$\begin{aligned}\dot{\tilde{\rho}} = &-i\Delta[a^\dagger a,\tilde{\rho}] + \frac{\gamma}{2}(2a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho} - \tilde{\rho}a^\dagger a)\\ &+\gamma\bar{n}(a\tilde{\rho}a^\dagger + a^\dagger\tilde{\rho}a - a^\dagger a\tilde{\rho} - \tilde{\rho}aa^\dagger).\end{aligned} \tag{1.71}$$

Here $\tilde{\rho}$ is still in the interaction picture. To transform back to the Schrödinger picture we use (1.26) to obtain

$$\dot{\rho} = \frac{1}{i\hbar}[H_S,\rho] + e^{-(i/\hbar)H_S t}\dot{\tilde{\rho}}\,e^{(i/\hbar)H_S t}. \tag{1.72}$$

With $H_S = \hbar\omega_0 a^\dagger a$, we substitute for $\dot{\tilde{\rho}}$ and use (1.26) and (1.40) to write, for example,

$$\begin{aligned}e^{-i\omega_0 a^\dagger a t}a\tilde{\rho}a^\dagger e^{i\omega_0 a^\dagger a t} &= e^{-i\omega_0 a^\dagger a t}a\left(e^{i\omega_0 a^\dagger a t}\rho e^{-i\omega_0 a^\dagger a t}\right)a^\dagger e^{i\omega_0 a^\dagger a t}\\ &= \left(e^{-i\omega_0 a^\dagger a t}a e^{i\omega_0 a^\dagger a t}\right)\rho\left(e^{-i\omega_0 a^\dagger a t}a^\dagger e^{i\omega_0 a^\dagger a t}\right)\\ &= a\rho a^\dagger.\end{aligned}$$

Each term can be treated similarly. We arrive at the *master equation for the damped harmonic oscillator*

$$\begin{aligned}\dot{\rho} = &-i\omega_0'[a^\dagger a,\rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\\ &+\gamma\bar{n}(a\rho a^\dagger + a^\dagger\rho a - a^\dagger a\rho - \rho aa^\dagger),\end{aligned} \tag{1.73}$$

where

$$\omega_0' \equiv \omega_0 + \Delta. \tag{1.74}$$

Note 1.3 An alternate, more compact, writing of the master equation (1.73) may be given in the form

$$\begin{aligned}\dot{\rho} = &-i\omega_0'[a^\dagger a,\rho] + \frac{\gamma}{2}([a,\rho a^\dagger] + [a\rho,a^\dagger])\\ &+\frac{\gamma}{2}\bar{n}([a\rho,a^\dagger] + [a^\dagger,\rho a]).\end{aligned} \tag{1.75}$$

In both this form and (1.73) the damping terms are grouped according to whether they are propotional to $\bar{n}$ or not. This is a natural grouping from the point of view of the phase-space representations commonly used in quantum optics which we meet in Chaps. 3 and 4 [see (3.47), for example, where the terms proportional and not proportional to $\bar{n}$ have distinct physical interpretations]. Nowadays, it is more usual to group the terms so that the Lindblad form of the master equation is explicit [1.12], writing

$$\begin{aligned}\dot{\rho} = &-i\omega_0'[a^\dagger a,\rho] + \frac{\gamma}{2}(\bar{n}+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\\ &+\frac{\gamma}{2}\bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger).\end{aligned} \tag{1.76}$$

Here the physical interpretation follows from the rate equations satisfied by the probabilities $p_n \equiv \langle n|\rho|n\rangle$ for the oscillator to be found in its nth energy eigenstate:

$$\begin{aligned}\dot{p}_n = &\gamma(\bar{n}+1)(n+1)p_{n+1} - \gamma\bar{n}np_n\\ &+\gamma\bar{n}np_{n-1} - \gamma\bar{n}(n+1)p_n.\end{aligned} \tag{1.77}$$

The terms on the right-hand side of (1.77) describe transition rates into and out of the nth energy level (see Fig. 7.4) and originate, respectively, in the terms proportional to $2a\rho a^\dagger$, $-(a^\dagger a\rho + \rho a^\dagger a)$, $2a^\dagger \rho a$, and $-(aa^\dagger\rho + \rho aa^\dagger)$ in (1.76) [also see the discussion below (2.27) and (2.36d)].

Note 1.4 There is a large literature on the treatment of dissipative quantum systems using semigroups, from which the work of Lindblad on the form of the generator for physical semigroup dynamics [1.12] is a result of particular relevance to quantum optics; thus, the master equations we met in this book are all of Lindblad form. The foundational work of Davies [1.13] has also been influential in quantum optics, particularly in relation to the theory of photon counting [1.14]. We will have more to say about this topic when we discuss quantum trajectories in Volume II (Chaps. 15 and 16). More generally, the orientation in the literature on semigroups is towards the proof of rigorous mathematical results and hence the connections to quantum optics applications are somewhat indirect.

1.4.2 Some Limitations

Equation (1.73) is one of the central equations for future applications. Before proceeding we should note its limitations as a general equation for the damped harmonic oscillator.

First, it is derived in the rotating-wave approximation (R.W.A.). We expect this to be a good approximation for oscillators at optical frequencies [1.15], but for low frequency oscillators (strong damping, where the decay time approaches the oscillator period) we would not expect the R.W.A. to work well. In fact, even at optical frequencies the R.W.A. brings one notable inaccuracy. The frequency shift Δ in (1.74) is small, and generally neglected. However, in the example of the damped two-level atom this is the Lamb shift, and it is therefore of fundamental importance. Of course, an accurate calculation of the Lamb shift must include many things that we do not discuss – for example, relativistic effects. Nevertheless, it is as well to know that the (two-level) nonrelativistic contribution to the Lamb shift is not obtained correctly when the master equation is derived using the rotating-wave approximation. A derivation that does not use the R.W.A. is quite straightforward and proceeds along the same lines as the calculation in Sect. 1.4.1. The details are given by Agarwal [1.16, 1.17], who, in Ref. [1.17] in particular, discusses the question of the frequency shift.

Secondly, (1.73) is not valid at low temperatures. At sufficiently low temperatures the reservoir correlation functions can no longer be treated as δ-functions. There is quite an active interest in this low temperature regime. Discussions can be found in recent papers by Caldeira and Leggett [1.3], Lindenberg and West [1.18], and Grabert et al. [1.19].

1.4.3 Expectation Values and Commutation Relations

Let us make some simple checks to see if (1.73) predicts the behavior we expect from a damped harmonic oscillator. Since we have formulated our theory in the Schrödinger picture, we cannot obtain solutions for the operators themselves, but only for their expectation values. For example, if we multiply (1.73) on the left by a and take the trace (over the system S) we obtain an equation for $\langle a\rangle = \mathrm{tr}(a\rho)$:

$$\begin{aligned}\langle \dot a\rangle &= -i\omega_0\,\mathrm{tr}(aa^\dagger a\rho - a\rho a^\dagger a) + \frac{\gamma}{2}\,\mathrm{tr}(2a^2\rho a^\dagger - aa^\dagger a\rho - a\rho a^\dagger a)\\ &\quad + \gamma\bar n\,\mathrm{tr}(a^2\rho a^\dagger + aa^\dagger\rho a - aa^\dagger a\rho - a\rho aa^\dagger)\\ &= -i\omega_0\,\mathrm{tr}\big[(aa^\dagger - a^\dagger a)a\rho\big] + \frac{\gamma}{2}\,\mathrm{tr}\Big[(a^\dagger a - aa^\dagger)a\rho\Big]\\ &\quad + \gamma\bar n\,\mathrm{tr}\big[(a^\dagger a - aa^\dagger)a\rho + a(aa^\dagger - a^\dagger a)\rho\big]\\ &= -\Big(\frac{\gamma}{2} + i\omega_0\Big)\langle a\rangle, \end{aligned} \tag{1.78}$$

where we have used the cyclic property of the trace and the boson commutation relation (1.10). From now on we assume that the frequency shift Δ is included in the resonance frequency of the oscillator and do not distinguish ω_0' from ω_0. Equation (1.78) correctly describes the damped mean oscillator amplitude.

As a second example consider $\langle\hat n\rangle = \langle a^\dagger a\rangle$:

$$\begin{aligned}\langle\dot{\hat n}\rangle &= -i\omega_0\,\mathrm{tr}(a^\dagger aa^\dagger a\rho - a^\dagger a\rho a^\dagger a) + \frac{\gamma}{2}\,\mathrm{tr}(2a^\dagger a^2\rho a^\dagger - a^\dagger aa^\dagger a\rho\\ &\quad - a^\dagger a\rho a^\dagger a) + \gamma\bar n\,\mathrm{tr}(a^\dagger a^2\rho a^\dagger + a^\dagger aa^\dagger\rho a - a^\dagger aa^\dagger a\rho - a^\dagger a\rho aa^\dagger)\\ &= \gamma\,\mathrm{tr}\big[a^{\dagger 2}a^2\rho - (a^\dagger a)^2\rho\big]\\ &\quad + \gamma\bar n\,\mathrm{tr}\big[a^{\dagger 2}a^2\rho + (aa^\dagger)^2\rho - (a^\dagger a)^2\rho - aa^{\dagger 2}a\rho\big]\\ &= -\gamma(\langle\hat n\rangle - \bar n), \end{aligned} \tag{1.79}$$

with the solution

$$\langle\hat n(t)\rangle = \langle\hat n(0)\rangle e^{-\gamma t} + \bar n(1 - e^{-\gamma t}). \tag{1.80}$$

Notice how thermal fluctuations are fed into the oscillator from the reservoir; the mean energy does not decay to zero but to the mean energy for an oscillator with frequency ω_0 in thermal equilibrium at temperature T.

Exercise 1.2 Show that the thermal equilibrium density operator

$$\rho_{\mathrm{eq}} = \frac{e^{-H_S/k_BT}}{\mathrm{tr}\Big(e^{-H_S/k_BT}\Big)} = \frac{e^{-\hbar\omega_0 a^\dagger a/k_BT}}{1 - e^{-\hbar\omega_0/k_BT}}$$

satisfies (1.73) in the steady state.

As a final observation we note that the boson commutation relation is preserved in time – at least in the mean, which is all we can say in the Schrödinger picture. Using the initial time commutator we find

$$\langle [a, a^\dagger](t)\rangle = \mathrm{tr}\{[a, a^\dagger]\rho(t)\} = \mathrm{tr}\{\rho(t)\} = 1;$$

it is readily shown that (1.73) preserves the trace of the density operator.

1.5 Two-Time Averages and the Quantum Regression Theorem

We have developed a formalism which allows us, in principle, to solve for the density operator (reduced density operator) for a system interacting with a reservoir. From this density operator we can obtain time-dependent expectation values for any operator acting in the Hilbert space of the system S. What, however, about products of operators evaluated at two different times? Of particular interest, for example, will be the first-order and second-order correlation functions of the electromagnetic field. For a single mode these are given by

$$G^{(1)}(t, t+\tau) \propto \langle a^\dagger(t)a(t+\tau)\rangle,$$
$$G^{(2)}(t, t+\tau) \propto \langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle.$$

The first-order correlation function is required for calculating the spectrum of the field. The second-order correlation function gives information about the photon statistics and describes photon bunching and antibunching.

Note 1.4 It may seem a strange talking about the spectrum of a single mode field since we normally associate a single mode with a single frequency. Here we are dealing, however, with what should more correctly be called a quasimode – a mode defined in a *lossy* optical cavity, which therefore has a finite linewidth.

Clearly, averages involving two times cannot be calculated directly from the master equation – at least, not without a little extra thought. We need to return to the microscopic picture of system plus reservoir. At this level two-time averages are defined in the usual way in the Heisenberg representation. Our objective, then, is to derive a relationship that allows us to calculate these averages at the macroscopic level using the master equation for the reduced density operator alone; thus, in some approximate way we wish to carry out the trace over reservoir variables explicitly, as we did in deriving the master equation itself. The result we obtain is known as the quantum

regression theorem and is attributed to Lax [1.20, 1.21]. We will not follow Lax in detail, but our method is fundamentally the same as his.

1.5.1 Formal Results

Recall our microscopic formulation of system S coupled to reservoir R. The Hamiltonian for the composite system $S \oplus R$ takes the form given in (1.16). The density operator is designated $\chi(t)$ and satisfies Schrödinger's equation (1.19). Our derivation of the master equation has given us an equation for the reduced density operator (1.17), which we will now write formally as

$$\dot{\rho} = \mathcal{L}\rho; \tag{1.81}$$

$\mathcal{L}$ is a generalized Liouvillian, a "superoperator" in the language of the Brussells-Austin group [1.22]; $\mathcal{L}$ operates on operators rather than on states. For the damped harmonic oscillator, from (1.73), the action of $\mathcal{L}$ on an arbitrary operator $\hat{O}$ is defined by the equation

$$\mathcal{L}\hat{O} \equiv -i\omega_0[a^\dagger a, \hat{O}] + \frac{\gamma}{2}(2a\hat{O}a^\dagger - a^\dagger a\hat{O} - \hat{O}a^\dagger a)$$
$$+ \gamma\bar{n}(a\hat{O}a^\dagger + a^\dagger\hat{O}a - a^\dagger a\hat{O} - \hat{O}aa^\dagger). \tag{1.82}$$

Within the microscopic formalism multi-time averages are straightforwardly defined in the Heisenberg picture. In particular, the average of a product of operators evaluated at two different times is given by

$$\langle \hat{O}_1(t)\hat{O}_2(t')\rangle = \mathrm{tr}_{S\oplus R}[\chi(0)\hat{O}_1(t)\hat{O}_2(t')], \tag{1.83}$$

where $\hat{O}_1$ and $\hat{O}_2$ are any two system operators. These operators satisfy the Heisenberg equations of motion

$$\dot{\hat{O}}_1 = \frac{1}{i\hbar}[\hat{O}_1, H], \tag{1.84a}$$
$$\dot{\hat{O}}_2 = \frac{1}{i\hbar}[\hat{O}_2, H], \tag{1.84b}$$

with the formal solutions

$$\hat{O}_1(t) = e^{(i/\hbar)Ht}\hat{O}_1(0)e^{-(i/\hbar)Ht}, \tag{1.85a}$$
$$\hat{O}_2(t') = e^{(i/\hbar)Ht'}\hat{O}_2(0)e^{-(i/\hbar)Ht'}. \tag{1.85b}$$

From (1.19), the formal solution for χ gives

$$\chi(0) = e^{(i/\hbar)Ht}\chi(t)e^{-(i/\hbar)Ht}. \tag{1.86}$$

We substitute these formal solutions into (1.83) and use the cyclic property of the trace to obtain

$$\begin{aligned}\langle\hat{O}_1(t)\hat{O}_2(t')\rangle &= \mathrm{tr}_{S\oplus R}\Big[e^{(i/\hbar)Ht}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\hat{O}_2(0)e^{-(i/\hbar)Ht'}\Big]\\ &= \mathrm{tr}_{S\oplus R}\Big[\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\\ &= \mathrm{tr}_S\Big\{\hat{O}_2(0)\mathrm{tr}_R\Big[e^{-(i/\hbar)H(t'-t)}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\Big\}.\end{aligned} \tag{1.87}$$

In the final step we have used the fact that $\hat{O}_2$ is an operator in the Hilbert space of S alone.

We now specialize to the case $t' \geq t$ and define

$$\tau \equiv t' - t, \tag{1.88}$$

$$\chi_{\hat{O}_1}(\tau) \equiv e^{-(i/\hbar)H\tau}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H\tau}. \tag{1.89}$$

Clearly, $\chi_{\hat{O}_1}$ satisfies the equation

$$\frac{d\chi_{\hat{O}_1}}{d\tau} = \frac{1}{i\hbar}\Big[H, \chi_{\hat{O}_1}\Big] \tag{1.90}$$

with

$$\chi_{\hat{O}_1}(0) = \chi(t)\hat{O}_1(0). \tag{1.91}$$

If we are to eliminate explicit reference to the reservoir in (1.87), we need to evaluate the reservoir trace over $\chi_{\hat{O}_1}(\tau)$ to obtain the reduced operator

$$\rho_{\hat{O}_1}(\tau) \equiv \mathrm{tr}_R\Big[\chi_{\hat{O}_1}(\tau)\Big], \tag{1.92}$$

where

$$\rho_{\hat{O}_1}(0) = \mathrm{tr}_R[\chi(t)\hat{O}_1(0)] = \mathrm{tr}_R[\chi(t)]\hat{O}_1(0) = \rho(t)\hat{O}_1(0); \tag{1.93}$$

notice that $\rho_{\hat{O}_1}(\tau)$ is just the term $\mathrm{tr}_R[\cdots]$ inside the curly brackets in (1.87). If we then assume that $\chi(t)$ factorizes as $\rho(t)R_0$, in the spirit of (1.29), from (1.91) and (1.93) we can write

$$\chi_{\hat{O}_1}(0) = R_0[\rho(t)\hat{O}_1(0)] = R_0\,\rho_{\hat{O}_1}(0). \tag{1.94}$$

Equations (1.90), (1.92), and (1.94) are now equivalent to (1.19), (1.17), and (1.25) – namely, to the starting equations in our derivation of the master equation. We can find an equation for $\rho_{\hat{O}_1}(\tau)$ in the Born-Markov approximation following a completely analogous course to that followed in Sects. 1.3 and 1.4. Since (1.19) and (1.90) contain the same Hamiltonian H, using the formal notation of (1.81), we arrive at the equation

$$\frac{d\rho_{\hat{O}_1}}{d\tau} = \mathcal{L}\rho_{\hat{O}_1}, \tag{1.95}$$

with solution

$$\rho_{\hat{O}_1}(\tau) = e^{\mathcal{L}\tau}\Big[\rho_{\hat{O}_1}(0)\Big] = e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]. \tag{1.96}$$

When we substitute for $\rho_{\hat{O}_1}(\tau)$ in (1.87), we have ($\tau \geq 0$)

$$\langle\hat{O}_1(t)\hat{O}_2(t+\tau)\rangle = \mathrm{tr}_S\{\hat{O}_2(0)e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]\}. \tag{1.97}$$

Exercise 1.3 Follow the same procedure to obtain ($\tau \geq 0$)

$$\langle\hat{O}_1(t+\tau)\hat{O}_2(t)\rangle = \mathrm{tr}_S\{\hat{O}_1(0)e^{\mathcal{L}\tau}[\hat{O}_2(0)\rho(t)]\}. \tag{1.98}$$

Equations (1.97) and (1.98) give formal statements of the *quantum regression theorem* for two-time averages. To calculate a correlation function $\langle\hat{O}_1(t)\hat{O}_2(t')\hat{O}_3(t)\rangle$ we cannot use (1.97) and (1.98) because noncommuting operators do not allow the reordering necessary to bring $\hat{O}_1(t)$ next to $\hat{O}_3(t)$. We may, however, generalize the approach taken above. Specifically, we have

$$\begin{aligned}&\langle\hat{O}_1(t)\hat{O}_2(t')\hat{O}_3(t)\rangle\\ &\quad= \mathrm{tr}_{S\oplus R}\Big[e^{(i/\hbar)Ht}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)}\\ &\qquad\times\hat{O}_3(0)e^{-(i/\hbar)Ht}\Big]\\ &\quad= \mathrm{tr}_{S\oplus R}\Big[\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\\ &\quad= \mathrm{tr}_S\Big\{\hat{O}_2(0)\mathrm{tr}_R\Big[e^{-(i/\hbar)H(t'-t)}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\Big\}.\end{aligned} \tag{1.99}$$

Defining

$$\chi_{\hat{O}_3\hat{O}_1}(\tau) \equiv e^{-(i/\hbar)H\tau}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H\tau} \tag{1.100}$$

and

$$\rho_{\hat{O}_3\hat{O}_1}(\tau) \equiv \mathrm{tr}_R\Big[\chi_{\hat{O}_3\hat{O}_1}(\tau)\Big] \tag{1.101}$$

as analogs of (1.89) and (1.92), we can proceed as before to the result ($\tau \geq 0$)

$$\langle\hat{O}_1(t)\hat{O}_2(t+\tau)\hat{O}_3(t)\rangle = \mathrm{tr}_S\{\hat{O}_2(0)e^{\mathcal{L}\tau}[\hat{O}_3(0)\rho(t)\hat{O}_1(0)]\}. \tag{1.102}$$

Equations (1.97) and (1.98) are, in fact, just special cases of (1.102) with either $\hat{O}_1(t)$ or $\hat{O}_3(t)$ set equal to the unit operator.

1.5.2 Quantum Regression Theorem for a Complete Set of Operators

It is possible to work directly with the rather formal expressions derived above. The formal expressions can also be reduced, however, to a more familiar form [1.20], which is often more convenient for doing calculations. Essentially, we will find that the equations of motion for expectation values of system operators (one-time averages) are also the equations of motion for correlation functions (two-time averages).

We begin by assuming that there exists a complete set of system operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$, in the following sense: that for an arbitrary operator $\hat{O}$, and for each $\hat{A}_\mu$,

$$\mathrm{tr}_S[\hat{A}_\mu(\mathcal{L}\hat{O})] = \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S(\hat{A}_\lambda\hat{O}), \tag{1.103}$$

where the $M_{\mu\lambda}$ are constants. In particular, from this it follows that

$$\begin{aligned}\langle\dot{\hat{A}}_\mu\rangle = \mathrm{tr}_S(\hat{A}_\mu\dot{\rho}) &= \mathrm{tr}_S[\hat{A}_\mu(\mathcal{L}\rho)] \\ &= \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S(\hat{A}_\lambda\rho) \\ &= \sum_\lambda M_{\mu\lambda}\langle\hat{A}_\lambda\rangle. \end{aligned}\tag{1.104}$$

Thus, expectation values $\langle\hat{A}_\mu\rangle$, $\mu = 1, 2, \ldots$, obey a coupled set of linear equations with the evolution matrix $\boldsymbol{M}$ defined by the $M_{\mu\lambda}$ that appear in (1.103). In vector notation,

$$\langle\dot{\hat{\boldsymbol{A}}}\rangle = \boldsymbol{M}\langle\hat{\boldsymbol{A}}\rangle, \tag{1.105}$$

where $\hat{\boldsymbol{A}}$ is the column vector of operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$. Now, using (1.97) and (1.103) ($\tau \geq 0$):

$$\begin{aligned}\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle &= \mathrm{tr}_S\{\hat{A}_\mu(0)(\mathcal{L}e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)])\} \\ &= \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S\{\hat{A}_\lambda(0)e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]\} \\ &= \sum_\lambda M_{\mu\lambda}\langle\hat{O}_1(t)\hat{A}_\lambda(t+\tau)\rangle, \end{aligned}\tag{1.106}$$

or,

$$\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\rangle = \boldsymbol{M}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\rangle, \tag{1.107}$$

where $\hat{O}_1$ can be any system operator, not necessarily one of the $\hat{A}_\mu$. This result is just what would be obtained by removing the angular brackets from (1.105) (written with $t \to t+\tau$, and $\cdot \equiv d/dt \to d/d\tau$), multiplying on the left by $\hat{O}_1(t)$, and then replacing the angular brackets. Hence, for each

operator $\hat{O}_1$, the set of correlation functions $\langle\hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle$, $\mu = 1, 2, \ldots$, with $\tau \geq 0$, satisfies the same equations (as functions of τ) as do the averages $\langle\hat{A}_\mu(t+\tau)\rangle$. This is perhaps the more familiar statement of the *quantum regression theorem*.

Exercise 1.4 For $\tau \geq 0$ show that

$$\frac{d}{d\tau}\langle\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle = \boldsymbol{M}\langle\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle. \tag{1.108}$$

Thus, we can also multiply (1.105) on the right by $\hat{O}_2(t)$, inside the average. Also show that

$$\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle = \boldsymbol{M}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle. \tag{1.109}$$

It may appear that this form of the quantum regression theorem is quite restricted, since its derivation relies on the existence of a set of operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$, for which (1.103) holds. We can show that this is always so, however, if a discrete basis $|n\rangle$, $n = 1, 2, \ldots$, exists; although, in general, the complete set of operators may be very large. Consider the operators

$$\hat{A}_\mu = \hat{A}_{nm} \equiv |n\rangle\langle m|. \tag{1.110}$$

Then

$$\begin{aligned}\mathrm{tr}_S[\hat{A}_{nm}(\mathcal{L}\hat{O})] &= \mathrm{tr}_S[|n\rangle\langle m|(\mathcal{L}\hat{O})] \\ &= \langle m|(\mathcal{L}\hat{O})|n\rangle \\ &= \langle m|\left(\mathcal{L}\sum_{n',m'}|n'\rangle\langle m'|\langle n'|\hat{O}|m'\rangle\right)|n\rangle \\ &= \sum_{n',m'}\langle m|(\mathcal{L}|n'\rangle\langle m'|)|n\rangle\langle n'|\hat{O}|m'\rangle \\ &= \sum_{n',m'}\langle m|(\mathcal{L}|n'\rangle\langle m'|)|n\rangle\,\mathrm{tr}_S(|m'\rangle\langle n'|\hat{O}) \\ &= \sum_{n',m'}M_{nm;n'm'}\,\mathrm{tr}_S(\hat{A}_{n'm'}\hat{O}), \end{aligned}\tag{1.111}$$

with

$$M_{nm;n'm'} \equiv \langle m|\Big(\mathcal{L}|m'\rangle\langle n'|\Big)|n\rangle. \tag{1.112}$$

In the last step we have interchanged the indices n' and m'. Equation (1.111) gives an expansion in the form of (1.103). The complete set of operators includes all the outer products $|n\rangle\langle m|$, $n = 1, 2, \ldots$, $m = 1, 2, \ldots$; this may be a small number of operators, a large but finite number of operators, or a double infinity of operators in the case of the Fock state basis.

1.5.3 Correlation Functions for the Damped Harmonic Oscillator

We will conclude our discussion of two-time averages with two simple examples based on the equations for expectation values for the damped harmonic oscillator [Eqs. (1.78) and (1.79)]. We first calculate the first-order correlation function $\langle a^\dagger(t)a(t+\tau)\rangle$. Equation (1.78) gives the equation of motion for the mean oscillator amplitude:

$$\langle \dot{a}\rangle = -\left(\frac{\gamma}{2}+i\omega_0\right)\langle a\rangle. \tag{1.113}$$

Then, with $\hat{A}_1 = a$ and $\hat{O}_1 = a^\dagger$, from (1.105) and (1.107), we may write

$$\frac{d}{d\tau}\langle a^\dagger(t)a(t+\tau)\rangle = -\left(\frac{\gamma}{2}+i\omega_0\right)\langle a^\dagger(t)a(t+\tau)\rangle. \tag{1.114}$$

Thus,

$$\begin{aligned}\langle a^\dagger(t)a(t+\tau)\rangle &= \langle a^\dagger(t)a(t)\rangle e^{-(\gamma/2+i\omega_0)\tau}\\ &= \left[\langle \hat{n}(0)\rangle e^{-\gamma t}+\bar{n}(1-e^{-\gamma t})\right]e^{-(\gamma/2+i\omega_0)\tau},\end{aligned} \tag{1.115}$$

where the last line follows from (1.80). If the oscillator describes a lossy cavity mode, in the long-time limit the Fourier transform of the first-order correlation function

$$\langle a^\dagger(0)a(\tau)\rangle_{ss} \equiv \lim_{t\to\infty}\langle a^\dagger(t)a(t+\tau)\rangle = \bar{n}e^{-(\gamma/2+i\omega_0)\tau} \tag{1.116}$$

gives the spectrum of the light at the cavity output. This is clearly a Lorentzian with width γ (full-width at half-maximum).

Note 1.5 This statement about the spectrum of the light at the cavity output is not strictly correct for the lossy cavity model as we have described it. The reason is that we have taken the environment outside the cavity to be in thermal equilibrium at temperature T (it is the environment that is modeled by the reservoir). Given this, the light detected in the cavity output will be a sum of transmitted light – light that passes from inside the cavity, through the cavity output mirror, into the environment – and thermal radiation reflected from the outside of the output mirror. Calculating the spectrum at the cavity output for this situation is more involved (Sect. 7.3.4). Physically, however, the result is clear; the spectrum must be a blackbody spectrum. The Lorentzian spectrum obtained from (1.116) would be observed, as filtered thermal radiation, for a cavity coupled to two reservoirs, one at temperature T and the other at zero temperature. If the bandwidth for coupling to the reservoir at temperature T is much larger than for coupling to the zero temperature reservoir, the master equation (1.73) is basically unchanged. Light emitted into the zero temperature reservoir then shows the Lorentzian spectrum obtained from the Fourier transform of (1.116).

For a second example we calculate the second-order correlation function $\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle = \langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle$. Writing (1.79) in the form

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{n}\rangle\\ \bar{n}\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma\\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle\hat{n}\rangle\\ \bar{n}\end{pmatrix}, \tag{1.117}$$

we set $\hat{A}_1 = \hat{n} = a^\dagger a$ and $\hat{A}_2 = \bar{n}$ (a constant). Then, from (1.105) and (1.109), with $\hat{O}_1 = a^\dagger$ and $\hat{O}_2 = a$,

$$\frac{d}{d\tau}\begin{pmatrix}\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle\\ \bar{n}\langle\hat{n}(t)\rangle\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma\\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle\\ \bar{n}\langle\hat{n}(t)\rangle\end{pmatrix}. \tag{1.118}$$

Thus,

$$\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle = \langle a^\dagger(t)\hat{n}(t)a(t)\rangle e^{-\gamma\tau} + \bar{n}\langle\hat{n}(t)\rangle(1-e^{-\gamma\tau}). \tag{1.119}$$

We obtained an expression for $\langle\hat{n}(t)\rangle$ in (1.80). The calculation of $\langle a^\dagger(t)\hat{n}(t)a(t)\rangle$ is left as an exercise:

Exercise 1.5 Derive an equation of motion for the expectation value $\langle a^\dagger(t)\hat{n}(t)a(t)\rangle = \langle a^{\dagger 2}(t)a^2(t)\rangle$ from the master equation (1.73) and show that

$$\begin{aligned}\langle a^\dagger(t)\hat{n}(t)a(t)\rangle = &\left[\langle\hat{n}^2(0)\rangle - \langle\hat{n}(0)\rangle\right]e^{-2\gamma t} + 2\bar{n}(1-e^{-\gamma t})\\ &\times\left[2\langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})\right].\end{aligned} \tag{1.120}$$

Now, substituting from (1.80) and (1.120) into (1.119),

$$\begin{aligned}&\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle\\ &\quad= \left\{\left[\langle\hat{n}^2(0)\rangle - \langle\hat{n}(0)\rangle\right]e^{-2\gamma t} + 2\bar{n}(1-e^{-\gamma t})\left[2\langle\hat{n}(0)\rangle e^{-\gamma t}\right.\right.\\ &\qquad\left.\left.+\bar{n}(1-e^{-\gamma t})\right]\right\}e^{-\gamma\tau} + \bar{n}\left[\langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})\right](1-e^{-\gamma\tau}).\end{aligned} \tag{1.121}$$

In the long-time limit, the second-order correlation function is

$$\begin{aligned}\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle_{ss} &\equiv \lim_{t\to\infty}\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle\\ &= \bar{n}^2(1+e^{-\gamma\tau}).\end{aligned} \tag{1.122}$$

This expression describes the well-known Hanbury-Brown-Twiss effect, or photon bunching, for thermal light [1.23]; at zero delay the correlation function has twice the value it has for long delays ($\gamma\tau \gg 1$).

Note 1.6 The correlation time, $1/\gamma$, in (1.122) holds for filtered thermal light in accord with the comments in Note 1.5.